MATH 212
Assignment 8

7.1.1

(a) <u>Way 1</u>

$$f(x) = e^{-(x+4)^2}$$

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+4)^2 - ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+4+\frac{1}{2}ik)^2 - \frac{1}{4}k^2 + 4ik}\, dx$$

$$= \frac{e^{-\frac{1}{4}k^2+4ik}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

$$= \frac{e^{-\frac{1}{4}k^2+4ik}}{\sqrt{2}}$$

$$(x+4)^2 + ikx$$
$$= x^2 + 16 + 8x + ikx$$
$$= x^2 + (8+ik)x + 16$$
$$= \left(x + \frac{8+ik}{2}\right)^2 + 16 - \left(\frac{8+ik}{2}\right)^2$$
$$= \left(x + 4 + \tfrac{1}{2}ik\right)^2 + 16 - \left(16 - \tfrac{1}{4}k^2 + 4ik\right)$$

<u>Way 2</u>

$$f(x) = e^{-(x+4)^2}$$

$$g(x) = e^{-x^2} \longrightarrow \tilde{\mathcal{F}}[g(x)] = \frac{e^{-k^2/4}}{\sqrt{2}}$$

By Theorem 7.4, $\tilde{\mathcal{F}}[f(x)] = e^{-ik(-4)}\, \tilde{\mathcal{F}}[g(x)]$

$$= \frac{e^{4ik}\, e^{-k^2/4}}{\sqrt{2}}$$

$$= \frac{e^{-\frac{1}{4}k^2+4ik}}{\sqrt{2}}$$

7.1.2

d) $f(x) = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ix} e^{ikx}\Big]_\alpha^\beta$

$= \frac{1}{\sqrt{2\pi}\, ix}\left[e^{i\beta x} - e^{-i\alpha x}\right]$ if $x \neq 0$

$f(x) = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta dk = \frac{\beta - \alpha}{\sqrt{2\pi}}$ if $x = 0$

7.1.2

b) $\tilde{F}[e^{-a|x|}] = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{k^2+a^2} \Rightarrow \tilde{F}[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \frac{1}{k^2+1}$

$\tilde{F}\left[\sqrt{\frac{2}{\pi}} \frac{1}{x^2+1}\right] = e^{-|-k|} = e^{-|k|}$

$\therefore \tilde{F}^{-1}[e^{-|k|}] = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{x^2+1}$ since $\frac{1}{x^2+1}$ is cont

7.1.3)

ⓐ $F[\text{sign}(x)] = -i\sqrt{\frac{2}{\pi}} \cdot \frac{1}{k}$

$\tilde{F}\left[i\sqrt{\frac{\pi}{2}}\,\text{sign}(x)\right] = \frac{1}{k}$

$\tilde{F}^{-1}\left[\frac{1}{K+a}\right] = \int_{-\infty}^{\infty} \frac{1}{K+a} e^{ikx}\, dK$

But $\tilde{F}\left[e^{-iax}\, i\sqrt{\frac{\pi}{2}}\,\text{sign}\,x\right] = \frac{1}{K+a}$

$\Rightarrow \tilde{F}^{-1}\left[\frac{1}{K+a}\right] = \begin{cases} e^{-iax}\, i\sqrt{\frac{\pi}{2}}\,\text{sign}\,x & |x|>0 \\ 0 & x=0 \end{cases}$

7.1.4) $\tilde{F}^{-1}\left[\frac{1}{K-a}\right] - \tilde{F}^{-1}\left[\frac{1}{K+a}\right] = \tilde{F}^{-1}\left[\frac{2a}{K^2-a^2}\right]$

$$\tilde{F}^{-1}\left[\frac{1}{K+a}\right] = \begin{cases} e^{-iax}\, i\sqrt{\frac{\pi}{2}}\, \text{sign}(x) & |x|>0 \\ 0 & x=0 \end{cases}$$

$$\tilde{F}^{-1}\left[\frac{1}{K-a}\right] = \begin{cases} e^{iax}\, i\sqrt{\frac{\pi}{2}}\, \text{sign}(x) & |x|>0 \\ 0 & x=0 \end{cases}$$

$$\tilde{F}^{-1}\left[\frac{2a}{K^2-a^2}\right] = \begin{cases} i\sqrt{\frac{\pi}{2}}\, \text{sign}(x)\left[e^{iax} - e^{-iax}\right] & |x|>0 \\ 0 & x=0 \end{cases}$$

$$= \begin{cases} -2\sqrt{\frac{\pi}{2}}\, \text{sign}(x)\sin ax & |x|>0 \\ 0 & x=0 \end{cases}$$

$$\tilde{F}^{-1}\left[\frac{1}{K^2-a^2}\right] = \begin{cases} -\sqrt{\frac{\pi}{2}}\, \frac{\text{sign}(x)}{a}\sin ax & |x|>0 \\ 0 & x=0 \end{cases}$$

7.1.5 (a) $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} e^{-ikx} dx$, $f(x) = e^{iwx}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix(k-w)} dx$$

$$\Rightarrow \hat{f}(k+w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} dx$$

$$= \sqrt{2\pi}\, \delta(k)$$

$$\Rightarrow \hat{f}(k) = \sqrt{2\pi}\, \delta(k-w)$$

(b) $$F[\cos wx] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[\frac{e^{iwx} + e^{-iwx}}{2} \right] dx$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-ikx} e^{iwx} dx + \int_{-\infty}^{\infty} e^{-ikx} e^{-iwx} dx \right]$$

(a) $$= \frac{1}{2} \left[\sqrt{2\pi}\, \delta(k-w) + \sqrt{2\pi}\, \delta(k+w) \right]$$

$$= \sqrt{\frac{\pi}{2}} \left[\delta(k-w) + \delta(k+w) \right]$$

$$F[\sin wx] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[\frac{e^{iwx} - e^{-iwx}}{2i} \right] dx$$

$$= \frac{1}{2i} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} e^{-ikx} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} e^{-ikx} dx \right]$$

$$= \frac{\sqrt{2\pi}}{2i} \left[\delta(k-w) - \delta(k+w) \right]$$

$$= -\sqrt{\frac{\pi}{2}}\, i \left[\delta(k-w) - \delta(k+w) \right]$$

7.1.6

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}\, dx - i \int_{-\infty}^{\infty} \frac{\sin kx}{x^2+a^2}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}\, dx - i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin kx}{x^2+a^2}\, dx$$

7.28 $$= \sqrt{\frac{\pi}{2}}\, \frac{e^{-a|k|}}{a}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}\, dx = \sqrt{\frac{\pi}{2}}\, \frac{e^{-a|k|}}{a}$$

$$\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}\, dx = \frac{\pi}{a}\, e^{-a|k|}$$

and $$\int_{-\infty}^{\infty} \frac{\sin kx}{x^2+a^2}\, dx = 0$$

7.1.11 Let $f(x)=g(x)+ih(x)$. By the linearity of Fourier Transform we have $\hat{\mathcal{F}}[f(x)]=\hat{\mathcal{F}}[g(x)+ih(x)]=\hat{\mathcal{F}}[g(x)]+i\hat{\mathcal{F}}[h(x)]$

$$\hat{f}(k)=\hat{g}(k)+i\hat{h}(k)$$

Let $\hat{f}_1(k)$ and $\hat{f}_2(k)$ be real valued functions s.t. $\hat{f}_1(k)=\mathrm{Re}[\hat{f}(k)]$ and $\hat{f}_2(k)=\mathrm{Im}[\hat{f}(k)]$; i.e $\hat{f}(k)=\hat{f}_1(k)+i\hat{f}_2(k)$. Is $\hat{f}_1(k)=\hat{g}(k)$ and $\hat{f}_2(k)=\hat{h}(k)$?

Proof: Let $\hat{g}_1(k)$, $\hat{g}_2(k)$, $\hat{h}_1(k)$ and $\hat{h}_2(k)$ be real valued functions st:

$$\hat{g}_1(k) = Re[\hat{g}(k)], \quad \hat{g}_2(k) = Im[\hat{g}(k)]$$
$$\hat{h}_1(k) = Re[\hat{h}(k)], \quad \hat{h}_2(k) = Im[\hat{h}(k)]$$

Then
$$\hat{f}(k) = \hat{g}_1(k) + i\,\hat{g}_2(k) + i\,\hat{h}_1(k) - \hat{h}_2(k)$$
$$= \hat{g}_1(k) - \hat{h}_2(k) + i\,(\hat{g}_2(k) + \hat{h}_1(k))$$
$$= \hat{f}_1(k) + i\,\hat{f}_2(k)$$

where $\hat{f}_1(k) = \hat{g}_1(k) - \hat{h}_2(k)$

and $\hat{f}_2(k) = \hat{g}_2(k) + \hat{h}_1(k)$

7.1.12) (a) Let $f(x)$ be even function

$$\hat{f}(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-y)\, e^{iky}\, dy$$

Let $y = -x$
$dy = -dx$

$$= \hat{f}(-k)$$

$\Rightarrow \hat{f}(k)$ is even

7.1.13) Proof of theorem 7.4

$$* \tilde{F}[f(x-\xi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X)\, e^{-ik(X+\xi)}\, dX \qquad X = x-\xi, \quad dX = dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X)\, e^{-ikX}\, e^{-ik\xi}\, dX$$

$$= e^{-ik\xi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X)\, e^{-ikX}\, dX$$

$$= e^{-ik\xi}\, \tilde{F}[f(x)] = e^{-ik\xi}\, \hat{f}(k)$$

$$* \tilde{F}[e^{i\xi x} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i(k-\xi)x}\, dx$$

$$= \hat{f}(k-\xi)$$

$\boxed{//}$

7.1.14) Proof of Dilation Theorem.

$$\tilde{F}[f(cx)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(cx)\, e^{-ikx}\, dx$$

Case 1: $c > 0$: Let $X = cx$, $dX = c\,dx$

$$\tilde{F}[f(cx)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X)\, e^{-i\frac{k}{c}X} \cdot \frac{1}{c}\, dX$$

$$= \frac{1}{c} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X)\, e^{-i\frac{k}{c}X}\, dX$$

$$= \frac{1}{c} \hat{f}\left(\frac{k}{c}\right) = \frac{1}{|c|} \hat{f}\left(\frac{k}{c}\right)$$

Case 2: $c < 0$: Let $X = cx$, $dX = c\,dx$

$$\tilde{F}[f(cx)] = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} f(X)\, e^{-i\frac{k}{c}X} \cdot \frac{1}{c}\, dX$$

$$= -\frac{1}{c} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(X)\, e^{-i\frac{k}{c}X}\, dX$$

$$= -\frac{1}{c} \hat{f}\left(\frac{k}{c}\right) = \frac{1}{|c|} \hat{f}\left(\frac{k}{c}\right)$$

$\boxed{//}$

7.2.1)

(a) $\tilde{F}[e^{-\frac{x^2}{2}}] \overset{\text{Table}}{\underset{\downarrow}{=}} \frac{e^{-k^2/4(\frac{1}{2})}}{\sqrt{2(\frac{1}{2})}} = \frac{e^{-k^2/2}}{\sqrt{1}} = e^{-k^2/2}$

(b) $\tilde{F}[-x e^{-\frac{x^2}{2}}] = ik\,\tilde{F}[e^{-\frac{x^2}{2}}]$

$= ik\, e^{-k^2/2}$ (since if $g(x) = e^{-\frac{x^2}{2}}$, then $g'(x) = -x e^{-\frac{x^2}{2}}$)

$\Rightarrow \tilde{F}[x e^{-\frac{x^2}{2}}] = -ik e^{-k^2/2}$

(c) $\tilde{F}[-x^2 e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}}] = ik\,\tilde{F}[x e^{-\frac{x^2}{2}}]$

$= ik[-ik e^{-k^2/2}]$

$= k^2 e^{-k^2/2}$ ($-x^2 e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}}$ is the derivative of $x e^{-x^2/2}$)

$-\tilde{F}[x^2 e^{-\frac{x^2}{2}}] + \tilde{F}[e^{-\frac{x^2}{2}}] = k^2 e^{-\frac{k^2}{2}}$

$\tilde{F}[x^2 e^{-\frac{x^2}{2}}] = e^{-k^2/2} - k^2 e^{-\frac{k^2}{2}}$

$= (1-k^2) e^{-\frac{k^2}{2}}$

<u>Or alternatively</u>:

(b) $\tilde{F}[x f(x)] = i \hat{f}'(k)$ where $f(x) = e^{-x^2/2}$

$\hat{f}(k) = e^{-k^2/2}$

$\tilde{F}[x e^{-\frac{x^2}{2}}] = i\left[-\frac{2}{2} k e^{-k^2/2}\right] = -ik e^{-k^2/2}$

(c) $\tilde{F}[x f(x)] = i \hat{f}'(k)$ where $f(x) = x e^{-\frac{x^2}{2}}$

$\hat{f}(k) = ik e^{-k^2/2}$

$\tilde{F}[x^2 e^{-\frac{x^2}{2}}] = i[-i e^{-\frac{k^2}{2}} + i k^2 e^{-\frac{k^2}{2}}]$

$= (1-k^2) e^{-\frac{k^2}{2}}$

2.4.20 (a) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,dx\,dy$

$$= \int_0^{2\pi}\int_0^{\infty} r e^{-ar^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2a} e^{-ar^2}\Big]_0^{\infty} d\theta = \int_0^{2\pi} \frac{1}{2a}\,d\theta$$

$$= \frac{1}{2a}\theta\Big]_0^{2\pi} = \frac{\pi}{a}$$

(b) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,dx\,dy$

$$= \int_{-\infty}^{\infty}\left[e^{-ay^2}\left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)dy\right]$$

$$= \int_{-\infty}^{\infty} e^{-ax^2}\,dx \int_{-\infty}^{\infty} e^{-ay^2}\,dy \overset{(a)}{=} \frac{\pi}{a}$$

$$\Rightarrow \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)^2 = \frac{\pi}{a}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

4.3.22

(a) $x^2y'' + 5xy' - 5y = 0$

$y = x^m \Rightarrow y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$m(m-1)x^m + 5mx^m - 5x^m = 0$

$m^2 - m + 5m - 5 = 0$

$m^2 + 4m - 5 = 0$

$m_1 = 1, \quad m_2 = -5$

$y = c_1 x^1 + c_2 x^{-5} = c_1 x + \frac{c_2}{x^5}$ c_1, c_2 ctes.

(d) $x^2y'' + xy' - 3y = 0$

$y = x^m \Rightarrow y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$m(m-1)x^m + mx^m - 3x^m = 0$

$m(m-1) + m - 3 = 0$

$m^2 - m + m - 3 = 0$

$m^2 - 3 = 0$

$m_1 = \sqrt{3}, \quad m_2 = -\sqrt{3}$

$y = c_1 x^{-\sqrt{3}} + c_2 x^{\sqrt{3}} = \frac{c_1}{x^{\sqrt{3}}} + c_2 x^{\sqrt{3}}$

4.3.26

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & r<1 \\ & -\pi \le \theta < \pi \\ u(1,\theta) = \cos^2\theta & -\pi \le \theta < \pi \end{cases}$$

$$u(r,\theta) = \frac{a_0}{2} + \sum_1^\infty [a_n \cos n\theta + b_n \sin n\theta] r^n$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2\theta \, d\theta = \frac{2}{\pi}\int_0^\pi \cos^2\theta \, d\theta$$

$$= \frac{2}{\pi}\int_0^\pi \frac{1+\cos 2\theta}{2} d\theta = \frac{2}{\pi}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\cdot\pi\right] = 1$$

$$u(0,0) = \frac{a_0}{2} = \frac{1}{2}$$

4.3.33

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_1^\infty (a_n r^n + b_n r^{-n})\cos n\theta + \sum_1^\infty (c_n r^n + d_n r^{-n}) \sin n\theta$$

$R_1 = 1$, $R_2 = 2$, $g_1(\theta) = 0$, $g_2(\theta) = h(\theta)$

(1)
$$\begin{cases} 2\pi(a_0 + b_0 \ln R_1) = \int_0^{2\pi} g_1(\theta)\, d\theta \\ 2\pi(a_0 + b_0 \ln R_2) = \int_0^{2\pi} g_2(\theta)\, d\theta \end{cases}$$

(2)
$$\begin{cases} a_n R_1^{\,n} + b_n R_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_1(\theta)\cos n\theta \, d\theta \\ a_n R_2^{\,n} + b_n R_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_2(\theta)\cos n\theta \, d\theta \end{cases} \quad n \ge 1$$

(3)
$$\begin{cases} c_n R_1^{\,n} + d_n R_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_1(\theta)\sin n\theta \, d\theta \\ c_n R_2^{\,n} + d_n R_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_2(\theta)\sin n\theta \, d\theta \end{cases} \quad n \ge 1$$

Solution (1)
$$\begin{cases} 2\pi(a_0) = \int_0^{2\pi} 0 \, d\theta \Rightarrow \boxed{a_0 = 0} \\ 2\pi(a_0 + b_0 \ln 2) = \int_0^{2\pi} g_2(\theta)\, d\theta \Rightarrow \boxed{b_0 = \frac{1}{2\pi \ln 2}\int_0^{2\pi} h(\theta)\, d\theta} \end{cases}$$

(2)
$$\begin{cases} a_n + b_n = \frac{1}{\pi}\int_0^{2\pi} 0 \, d\theta \Rightarrow a_n + b_n = 0 \Rightarrow a_n = -b_n \\ a_n 2^n + b_n \cdot 2^{-n} = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta \, d\theta \end{cases}$$

$$-b_n 2^n + b_n 2^{-n} = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta \, d\theta$$

$$b_n(2^{-n} - 2^n) = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta \, d\theta$$

$$\boxed{\begin{aligned} b_n &= \frac{1}{\pi(2^{-n}-2^n)}\int_0^{2\pi} h(\theta)\cos n\theta \, d\theta \\ a_n &= \frac{-1}{\pi(2^{-n}-2^n)}\int_0^{2\pi} h(\theta)\cos n\theta \, d\theta \end{aligned}}$$

(3)
$$\begin{cases} c_n + d_n = \frac{1}{\pi}\int_0^{2\pi} 0 \, d\theta \Rightarrow c_n = -d_n \\ c_n 2^n + d_n 2^{-n} = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin n\theta \, d\theta \end{cases}$$

$$d_n(2^{-n} - 2^n) = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin n\theta \, d\theta$$

$$\boxed{\begin{aligned} d_n &= \frac{1}{\pi(2^{-n}-2^n)}\int_0^{2\pi} h(\theta)\sin n\theta \, d\theta \\ c_n &= \frac{-1}{\pi(2^{-n}-2^n)}\int_0^{2\pi} h(\theta)\sin n\theta \, d\theta \end{aligned}}$$

4.3.34 - (a) $u(1,\theta) = 0$ $\quad 1 < r < 2$

$u(2,\theta) = 1$

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_1^\infty (a_n r^n + b_n r^{-n}) \cos n\theta + \sum_1^\infty (c_n r^n + d_n r^{-n}) \sin n\theta$$

$$\begin{cases} 2\pi(a_0 + 0) = \int_0^{2\pi} 0 \, d\theta \Rightarrow a_0 = 0 \\ 2\pi(a_0 + b_0 \ln 2) = \int_0^{2\pi} d\theta \Rightarrow b_0 \ln 2 = 1 \Rightarrow b_0 = \frac{1}{\ln 2} \end{cases}$$

$$\begin{cases} a_n + b_n = \frac{1}{\pi} \int_0^{2\pi} 0 \, d\theta \Rightarrow a_n = -b_n \\ a_n 2^n + b_n 2^{-n} = \frac{1}{\pi} \int_0^{2\pi} \cos n\theta \, d\theta = \frac{1}{n\pi} \sin n\theta \Big]_0^{2\pi} = 0 \end{cases}$$

$a_n 2^n - a_n 2^{-n} = 0$

$(2^n - 2^{-n}) a_n = 0$ $\quad n \geq 1$

$a_n = 0$

$b_n = 0$

$$\begin{cases} c_n + d_n = \frac{1}{\pi} \int_0^{2\pi} 0 \, d\theta \Rightarrow c_n = -d_n \\ c_n 2^n + d_n 2^{-n} = \frac{1}{\pi} \int_0^{2\pi} \sin n\theta \, d\theta = -\frac{1}{n\pi} \cos n\theta \Big]_0^{2\pi} = 0 \end{cases}$$

$(2^n - 2^{-n}) c_n = 0$

$n \geq 1$

$c_n = 0$

$d_n = 0$

$\therefore u(r,\theta) = \frac{\ln r}{\ln 2}$

4.3.34 (c) $u(1,\theta) = \sin 2\theta \qquad 1 < r < 2$

$u(2,\theta) = \cos 2\theta$

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_1^\infty (a_n r^n + b_n r^{-n}) \cos n\theta + \sum_1^\infty (c_n r^n + d_n r^{-n}) \sin n\theta$$

(1) $\begin{cases} 2\pi (a_0) = \int_0^{2\pi} \sin 2\theta \, d\theta \Rightarrow a_0 = 0 \\ 2\pi (a_0 + b_0 \ln 2) = \int_0^{2\pi} \cos 2\theta \, d\theta \Rightarrow b_0 = 0 \end{cases}$

(2) $n \geq 1$ $\begin{cases} a_n + b_n = \frac{1}{\pi} \int_0^{2\pi} \sin 2\theta \cos n\theta \, d\theta = 0 \Rightarrow a_n = -b_n \\ a_n 2^n + b_n 2^{-n} = \frac{1}{\pi} \int_0^{2\pi} \cos 2\theta \cos n\theta \, d\theta = \begin{cases} \frac{\pi}{\pi} = 1 & n = 2 \\ 0 & n \neq 2 \end{cases} \end{cases}$

$n = 2$: $a_2 \cdot 4 + b_2 \cdot \frac{1}{4} = 1$

$4a_2 + \frac{1}{4} b_2 = 1 \Rightarrow -4b_2 + \frac{1}{4} b_2 = 1 \Rightarrow -15 b_2 = 4$

$\Rightarrow b_2 = -\frac{4}{15}$

$a_2 = \frac{4}{15}$

$n \neq 2$: $a_n 2^n + b_n 2^{-n} = 0$

$(2^n - 2^{-n}) a_n = 0 \Rightarrow a_n = 0$

(3) $\begin{cases} c_n + d_n = \frac{1}{\pi} \int_0^{2\pi} \sin 2\theta \sin n\theta \, d\theta = \begin{cases} 1 & n = 2 \\ 0 & n \neq 2 \end{cases} \\ c_n \cdot 2^n + d_n \cdot 2^{-n} = \frac{1}{\pi} \int_0^{2\pi} \cos 2\theta \sin n\theta \, d\theta = 0 \end{cases}$

$\hookrightarrow 2^n c_n + 2^{-n} d_n = 0$

$n \neq 2$: $c_n + d_n = 0, \quad c_n = -d_n$

$2^n c_n - 2^{-n} c_n = 0 \Rightarrow c_n = 0 \Rightarrow d_n = 0$

$n = 2$: $c_2 + d_2 = 1 \Rightarrow c_2 = 1 - d_2$

$2^2 c_2 + \frac{1}{4} d_2 = 0 \Rightarrow 4(1 - d_2) + \frac{1}{4} d_2 = 0$

$16 - 16 d_2 + d_2 = 0 \Rightarrow d_2 = \frac{16}{15} \Rightarrow c_2 = -\frac{1}{15}$

$$u(r,\theta) = \left(\frac{4}{15} r^2 - \frac{4}{15 r^2}\right) \cos 2\theta + \left(-\frac{1}{15} r^2 + \frac{16}{15 r^2}\right) \sin 2\theta$$